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LETTER TO THE EDITOR

Relations connecting the scale transformation and Bäcklund transformation for the cylindrical Korteweg–de Vries equation

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Abstract. Relations $B_a = S^{-1}(a)BS(a)$ and $B_a^{-1} = S(-1)B_aS(-1)$, connecting the scale transformation (ST) and the Bäcklund transformation (BT) for the cylindrical Korteweg–de Vries equation, $u_t + 6uu_x + u_{xxx} + u/2t = 0$, are obtained. One analogous relation between the BT and translation in x is also considered.

In the study of solitons in multi-dimensional systems, one of the simplest, and thus most important, models is provided by the cylindrical Korteweg–de Vries equation first studied by Maxon and Viecelli (1974). They considered the propagation of radially ingoing acoustic waves of a plasma with cylindrical geometry. Under the assumptions of isothermal and stationary electrons, cold ions and small amplitude of the initial perturbation, they derived the cylindrical Korteweg–de Vries equation for the dimensionless ion fluid velocity written as

$$u_t + 6uu_x + u_{xxx} + u/2t = 0. \tag{1}$$

Here and in the following, a subscript represents partial differentiation. The analytical study of equation (1) has been carried out so far mainly via the inverse spectral transformation method by Calogero and Degasperis (1978). Very recently, a BT of equation (1) was obtained by Nimmo and Crighton (1981). In contrast to other known BT's this BT is unusual in that the integration of the BT with zero as one solution does not generate a single-soliton. In this letter we set up three relations connecting this BT with the scale transformation and the operation of translation. It turns out that to derive the conservation laws, nonlinear superposition formulae and the soliton solutions from the BT must make use of the arbitrary parameter contained in the BT; so it is helpful to set up such relations. The analogous relations for the sine-Gordon, Korteweg–de Vries and nonlinear Schrödinger equations have been obtained by Steudel (1975, 1980).

Let $u = w_x$. We rewrite equation (1) as

$$w_t + 3(w_x)^2 + w_{xxx} + w/2t = 0. \tag{2}$$

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A Bäcklund transformation for equation (2) is

$$B_a: \quad (w'+w)_x = (x+a)/6t - \tfrac{1}{2}(w'-w)^2, \tag{3a}$$

$$(w'-w)_t = (w'_{xx} - w_{xx})(w'-w) - 2(w'_x)^2 - 2w'_x w_x - 2w_x^2 - (w'-w)/2t, \tag{3b}$$

where w' and w are two solutions of equation (2) and a is an arbitrary parameter.

The scale transformation $S(\lambda)$ is defined as

$$S(\lambda): \qquad x \to \tilde{x} = \lambda^{-1}x, \qquad t \to \tilde{t} = \lambda^{-3}t, \qquad w \to \tilde{w} = \lambda w. \tag{4}$$

Obviously, equation (2) remains unchanged under the ST (4). Now we have:

Theorem 1.

$$B_a = S^{-1}(\lambda)B_{a/\lambda}S(\lambda), \tag{5}$$

and, in particular,

$$B_a = S^{-1}(a)B_1S(a), \tag{6}$$

where $B_{a/\lambda}$ and B_1 are the BT (3) with the parameter taking the values a/λ and 1, respectively.

Proof. Let $w_{(i)}$ $(i=0, 1)$ be two solutions of equation (2) and $w_{(1)}$ the result of $w_{(0)}$ under the BT (3). We use the formal notation

$$\begin{pmatrix} x \\ t \\ w_{(1)} \end{pmatrix} = B_a \begin{pmatrix} x \\ t \\ w_{(0)} \end{pmatrix}. \tag{7}$$

Using ST (4), we translate $w_{(i)}$ into $\tilde{w}_{(i)}$ $(i=0, 1)$, respectively, and write formally

$$\begin{pmatrix} \tilde{x} \\ \tilde{t} \\ \tilde{w}_{(i)} \end{pmatrix} = S(\lambda) \begin{pmatrix} x \\ t \\ w_{(i)} \end{pmatrix}, \qquad (i=0, 1). \tag{8}$$

It can easily be verified that

$$(\tilde{w}_{(1)} + \tilde{w}_{(0)})_{\tilde{x}} = (\tilde{x} + a/\lambda)/6\tilde{t} - \tfrac{1}{2}(\tilde{w}_{(1)} - \tilde{w}_{(0)})^2, \tag{9a}$$

$$\begin{aligned}(\tilde{w}_{(1)} - \tilde{w}_{(0)})_{\tilde{t}} = {} & (\tilde{w}_{(1)\tilde{x}\tilde{x}} - \tilde{w}_{(0)\tilde{x}\tilde{x}})(\tilde{w}_{(1)} - \tilde{w}_{(0)}) - 2(\tilde{w}_{(1)\tilde{x}})^2 - 2\tilde{w}_{(1)\tilde{x}}\tilde{w}_{(0)\tilde{x}} \\ & - 2(\tilde{w}_{(0)\tilde{x}})^2 - (2\tilde{t})^{-1}(\tilde{w}_{(1)} - \tilde{w}_{(0)}),\end{aligned} \tag{9b}$$

that is

$$\begin{pmatrix} \tilde{x} \\ \tilde{t} \\ \tilde{w}_{(1)} \end{pmatrix} = B_{a/\lambda} \begin{pmatrix} \tilde{x} \\ \tilde{t} \\ \tilde{w}_{(0)} \end{pmatrix}. \tag{10}$$

From (8) and (7), we have

$$S(\lambda)B_a \begin{pmatrix} x \\ t \\ w_{(0)} \end{pmatrix} = B_{a/\lambda}S(\lambda) \begin{pmatrix} x \\ t \\ w_{(0)} \end{pmatrix}; \tag{11}$$

therefore,

$$B_a = S^{-1}(\lambda)B_{a/\lambda}S(\lambda),$$

which is the conclusion (5), and the proof of (6) is immediate by taking $\lambda = a$ in equation (5).

Theorem 2.

$$B_a^{-1} = S(-1)B_aS(-1), \tag{12}$$

where $B_a^{-1} = (B_a)^{-1}$, the inverse transformation of the BT (3).

Proof. It is easy to see that $B_a^{-1} = B_{-a}$, hence formula (12) is true by equation (5).

Remark. The transformation $S(-1)$ is discrete. This transformation maps components of the connection $S^+(\lambda > 0)$ of the group of ST (4) on the component $S^-(\lambda < 0)$ and conversely.

Once more we specialise the BT (3) to $a = 0$, and it is easily seen that the relation

$$B_a = T^{-1}(a)B_0T(a) \tag{13}$$

holds, where $T(a)$ is the translation defined by

$$T(a): \qquad x \to \tilde{x} = x + a, \qquad t \to \tilde{t} = t, \qquad w \to \tilde{w} = w, \tag{14}$$

under which equation (2) is also invariant.

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